Introduction to Basic Electronic Circuits
non-linear circuits

Ph.D., D.Sc. Bogdan Pankiewicz - lecture, WETI PG room EA307

www.ue.eti.pg.gda.pl/~bpa/ec/

Lecture content

- Distortion and gain measures for non-inertial systems for harmonic excitation
- Differential bipolar pair
- Simple analogue multiplier using differential pair
- Gilbert multiplier
- Selected non-linear circuits

Literature:

1) U. Tieze, C. Schenk, E Gamm „Electronic circuits, handbook for design and applications”, Springer 2002.

2) R. L. Geiger, P. E. Allen, N. R. Strader, „VLSI design techniques for analog and digital circuits“, McGraw-Hill 1990.

3) P. E. Allen, D. R. Holberg, „ CMOS analog circuit design“, Sunders College Publishing, 1987.

4) P. R. Gray, R. G. Meyer, ”Analysis and design of analog integrated circuits”, John Wiley & Son, Inc. 1993.

5) P. Wambacq, W. Sansen, „Distortion analysis of analog integrated circuits”, Kluwer Academic Publishers, 1998.

Distortion and gain measures for non-inertial systems for harmonic excitation

For simplicity non-inertial non-linear circuit is assumed. Its transfer function is given by the polynomial:

$$v_{OUT} = f\left(v_{IN}\right) = K_0 + K_1 v_{IN} + K_2 v_{IN}^2 + K_3 v_{IN}^3$$

Gain an distortion for single harmonic excitation can be found by substitution:

$$v_{IN} = B\cos(\omega_B t)$$

$$\cos^2\alpha = \frac{1}{2} + \frac{1}{2}\cos(2\alpha)$$

Components of individual powers:

$$\cos^3\alpha = \frac{1}{4}\cos(3\alpha) + \frac{3}{4}\cos\alpha$$

$$K_1 v_{IN} = K_1 B\cos(\omega_B t)$$

$$K_2 {v_{IN}}^2 = K_2 B^2\cos^2(\omega_B t) = \frac{1}{2}K_2 B^2 + \frac{1}{2}K_2 B^2\cos(2\omega_B t)$$

$$K_3 {v_{IN}}^3 = K_3 B^3\cos^3(\omega_B t) = \frac{1}{4}K_3 B^3\cos(3\omega_B t) + \frac{3}{4}K_3 B^3\cos(\omega_B t)$$

Distortion and gain measures for non-inertial systems for harmonic excitation

$$v_{OUT} = f\left(B\cos(\omega_B t)\right) = K_0 + K_1 B\cos(\omega_B t) + \frac{1}{2}K_2 B^2 + \frac{1}{2}K_2 B^2\cos(2\omega_B t)$$

$$+\frac{1}{4}K_3 B^3\cos(3\omega_B t) + \frac{3}{4}K_3 B^3\cos(\omega_B t)$$

$$= K_0 + \frac{1}{2}K_2 B^2 + \left[K_1 B + \frac{3}{4}K_3 B^3\right]\cos(\omega_B t) + \frac{1}{2}K_2 B^2\cos(2\omega_B t) + \frac{1}{4}K_3 B^3\cos(3\omega_B t)$$

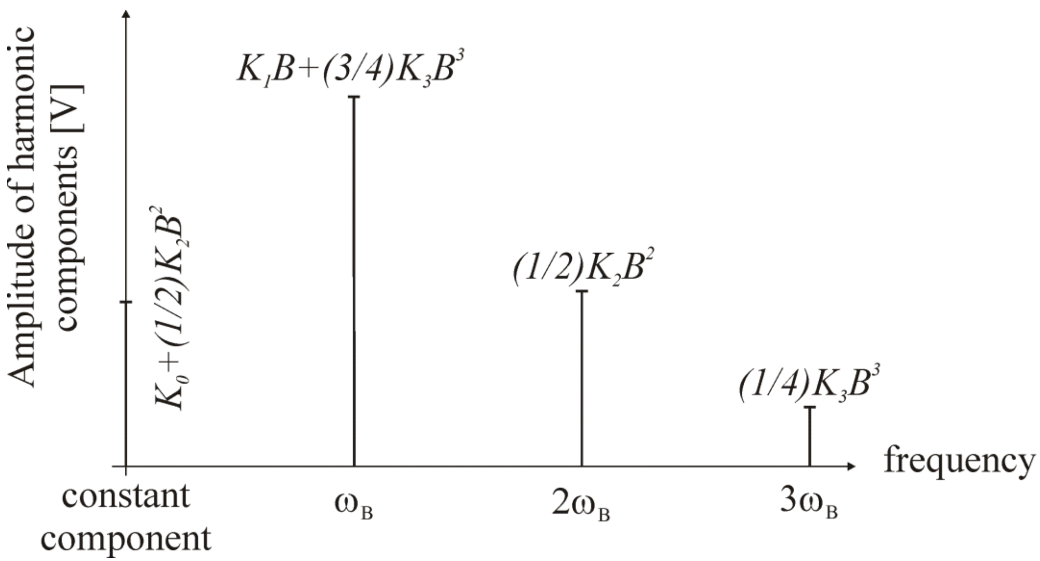


Fig. 1. Harmonic content with for harmonic excitation of non-inertial system described by 3rd degree polynomial.

Distortion and gain measures for non-inertial systems for harmonic excitation

- Individual powers of a polynomial result in harmonics with frequencies that are multiples of the power of a polynomial.
- The even powers of the polynomial also change the constant component of the output signal and make it dependent on the amplitude of the input signal.
- The amplitudes of individual subsequent harmonics are decreasing with the harmonic number.
- With slight distortion of the processing system, you can approximate its properties by limiting the describing function to the first few powers of the polynomial.

Bipolar differential pair

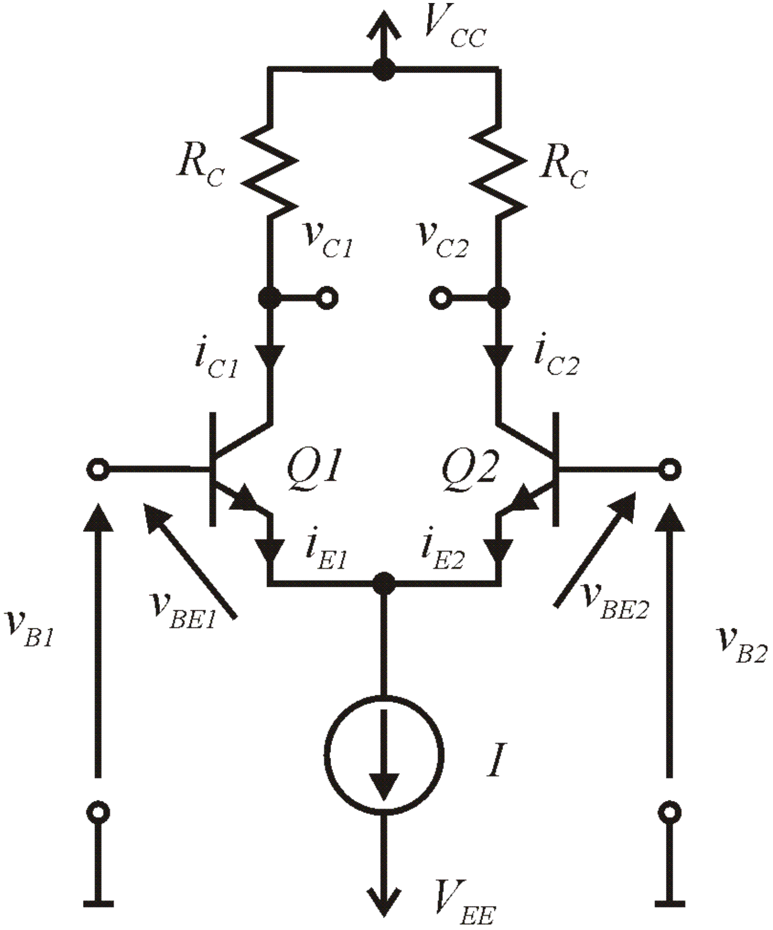


Fig. 2. Bipolar differential pair.

Emitter currents of transistors:

$$i_{E1} = \frac{I_S}{\alpha} e^{\frac{v_{B1} - v_E}{V_T}} \qquad i_{E2} = \frac{I_S}{\alpha} e^{\frac{v_{B2} - v_E}{V_T}}$$

hence:

$$\frac{i_{E1}}{i_{E2}} = \frac{\frac{I_S}{\alpha} e^{\frac{v_{B1} - v_E}{V_T}}}{\frac{I_S}{\alpha} e^{\frac{v_{B2} - v_E}{V_T}}} = e^{\frac{v_{B1} - v_{B2}}{V_T}} = e^{\frac{v_{ID}}{V_T}}$$

Together with Kirchhoff's current law, it gives the following equations:

$$\begin{cases} i_{E1} + i_{E2} = I \\ i_{E1} / i_{E2} = e^{v_{ID}/V_T} \end{cases}$$

The harmonic signal gain

Def: Gain of the harmonic wave can be defined as the ratio of the amplitude of the first harmonic output signal to the amplitude of the input signal:

$$A_H(B) = \frac{\text{1-st harmonic amplitude}}{\text{amplitude of input sine wave}} = \frac{\left[K_1 B + \frac{3}{4} K_3 B^3\right]}{B} = K_1 + \frac{3}{4} K_3 B^2$$

The above gain is not constant and depends on the amplitude of the input signal. Depending on the sign of the K_3 coefficient, both increase (expansion) and reduction (compression) of the gain can occur.

Bipolar differential pair, cont.

The solution gives the following results:

$$\begin{cases} i_{E1} = I\left(\dfrac{1}{1+e^{-v_{ID}/V_T}}\right) \\ i_{E2} = I\left(\dfrac{1}{1+e^{v_{ID}/V_T}}\right) \end{cases}$$

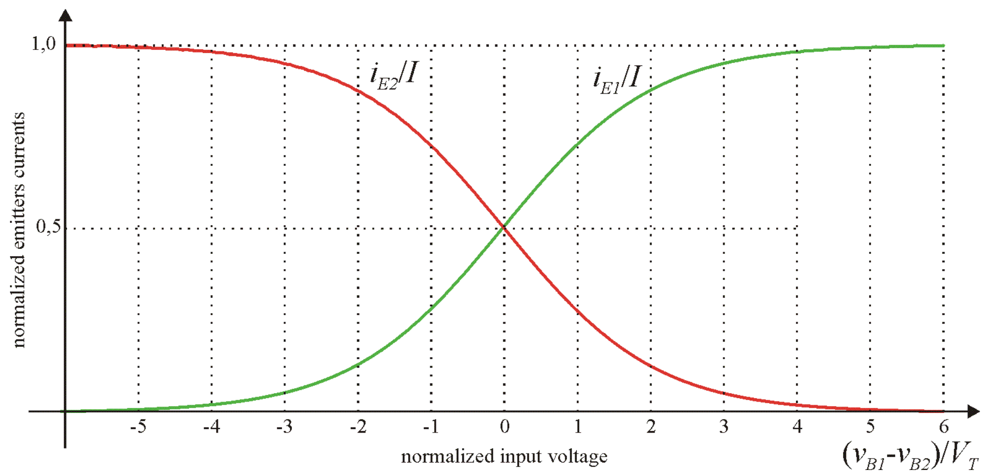


Fig. 3. Normalized emitter currents of bipolar differential pair.

Using substitution: $\dfrac{v_{ID}}{V_T} = x = 2y$ a changeable part of the emitter current can be calculated as:

$$i_{E1} - \frac{I}{2} = \frac{I}{1+e^{x}} - \frac{I}{2} = \frac{I}{2}\left(\frac{e^{x}-1}{e^{x}+1}\right) = \frac{I}{2}\left(\frac{e^{2y}-1}{e^{2y}+1}\right) = \frac{I}{2}\left(\frac{e^{2y-y}-e^{-y}}{e^{2y-y}+e^{-y}}\right)$$

$$= \frac{I}{2}\left(\frac{e^{y}-e^{-y}}{e^{y}+e^{-y}}\right) = \frac{I}{2}\tanh(y) = \frac{I}{2}\tanh\left(\frac{v_{ID}}{2V_T}\right)$$

and:

$$i_{E2} - \frac{I}{2} = -\frac{I}{2}\tanh(y) = -\frac{I}{2}\tanh\left(\frac{v_{ID}}{2V_T}\right)$$

Bipolar differential pair, cont.

The difference of emitter currents is therefore equal to:

$$i_{E1} - i_{E2} = I \tanh(y) = I \tanh\left(\frac{v_{ID}}{2V_T}\right)$$

The difference of collector currents will be equal:

$$i_{C1} - i_{C2} = \alpha I \tanh(y) = \alpha I \tanh\left(\frac{v_{ID}}{2V_T}\right)$$

If we treat current *I* as one of the processed signals, then we obtain the multiplication of the term ***tanh(x)*** and the current I. The term ***tanh(x)***, for a strong limitation of x << 1 (i.e. for $v_{ID} << 2V_T$) can be approximated by a linear function.

$$\tanh(x) \approx x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \ldots$$

So, if $v_{ID} << 2V_T$ then:

$$i_{C1} - i_{C2} \approx \alpha I \left(\frac{v_{ID}}{2V_T}\right)$$

NOTE: there are no even terms in the expansion of the output current into a power series – it results in reduced value of harmonic distortion.

Bipolar differential pair, cont.

We expand output current into a power series and limit to the 4th term:

$$i_{C1} - i_{C2} = \alpha I \tanh(y) = \alpha I \tanh\left(\frac{v_{ID}}{2V_T}\right) = \alpha I\left(\frac{v_{ID}}{2V_T} - \frac{1}{3}\left(\frac{v_{ID}}{2V_T}\right)^3 + \ldots\right) \approx \alpha I\left(\frac{v_{ID}}{2V_T} + \frac{v_{ID}^{\ 3}}{24V_T^{\ 3}}\right)$$

$$i_{C1} - i_{C2} = f(v_{ID}) = K_0 + K_1 v_{IN} + K_2 v_{IN}^2 + K_3 v_{IN}^3 + K_4 v_{IN}^4$$

$$K_0 = 0 \qquad K_1 = \frac{\alpha I}{2V_T} \qquad K_2 = 0 \qquad K_3 = -\frac{\alpha I}{24V_T^{\ 3}} \qquad K_4 = 0$$

$$\tanh(x) \approx x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \ldots$$

$$HD_2 \approx \frac{1}{2}\frac{K_2}{K_1}B = 0$$

$$HD_3 \approx \frac{1}{4}\frac{K_3}{K_1}B^2 = \frac{1}{4}\frac{2V_T}{24V_T^{\ 3}}B^2 = \frac{1}{48V_T^2}B^2 \Rightarrow THD \approx HD_3 \Rightarrow B = \sqrt{48V_T^2 THD}$$

For example, for THD <1%, the amplitude of the harmonic signal fed to the differential pair input must be:

$$B < \sqrt{48V_T^2 THD} = \sqrt{48 \cdot 25[\mathrm{mV}]^2\, 0{,}01} = 17{,}3[\mathrm{mV}]$$

Bipolar differential pair, cont.

The gain of harmonic signals, which is transconductance for a differential pair, will be equal to:

$$GM_H = K_1 + \frac{3}{4} K_3 B^2 = \frac{\alpha I}{2V_T} - \frac{3}{4}\frac{\alpha I}{24 V_T^3} B^2$$

Denoting as low signal transconductance: $gm = \frac{\alpha I}{2V_T}$

we get:

$$GM_H = gm\left(1 - \frac{1}{16 V_T^2} B^2\right)$$

The above calculations are valid as long as the higher order factors are not dominant, in order to calculate this we can apply the inequality:

$$\tanh(x) \approx x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \ldots \qquad \left|\frac{2x^5}{15}\right| << \left|-\frac{x^3}{3}\right| \Rightarrow |x| << \sqrt{5/2}$$

Because: $x = \frac{v_{ID}}{2V_T}$ so: $|v_{ID}| << 2V_T\sqrt{5/2} = 80[\mathrm{mV}]$

In practice, due to the quadratic relationship between terms of 5 and 3 order limitation $v_{ID} < 2V_T$ is enough.

Bipolar differential pair as a simple multiplier.

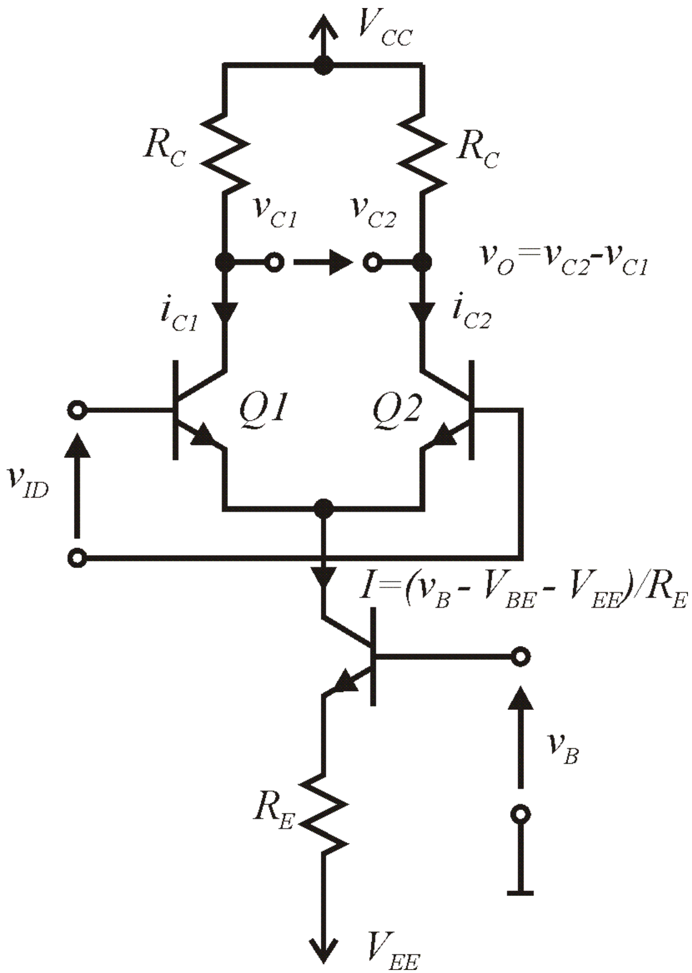


Fig. 4. Bipolar differential pair with current source implemented on Q3 transistor.

The difference of collector currents is equal to:

$$i_{C1}-i_{C2}=\alpha I\tanh\left(\frac{v_{ID}}{2V_T}\right)=\alpha\frac{v_B-V_{BE}-V_{EE}}{R_E}\tanh\left(\frac{v_{ID}}{2V_T}\right)$$

$$=\alpha\frac{v_B}{R_E}\tanh\left(\frac{v_{ID}}{2V_T}\right)+\alpha\frac{-V_{BE}-V_{EE}}{R_E}\tanh\left(\frac{v_{ID}}{2V_T}\right)$$

The output voltage is equal:

$$v_O=v_{C2}-v_{C1}=V_{CC}-i_{C2}R_C-\left(V_{CC}-i_{C1}R_C\right)$$

$$=i_{C1}R_C-i_{C2}R_C=\left(i_{C1}-i_{C2}\right)R_C$$

And so:

$$v_O=\alpha\frac{v_B}{R_E}\tanh\left(\frac{v_{ID}}{2V_T}\right)R_C$$ <- multiplication

$$+\alpha\frac{-V_{BE}-V_{EE}}{R_E}\tanh\left(\frac{v_{ID}}{2V_T}\right)R_C$$ <- gain

Gilbert multiplier cell

For simplicity, $\alpha = 1$ is assumed, so the collector currents are equal to the emitter currents.

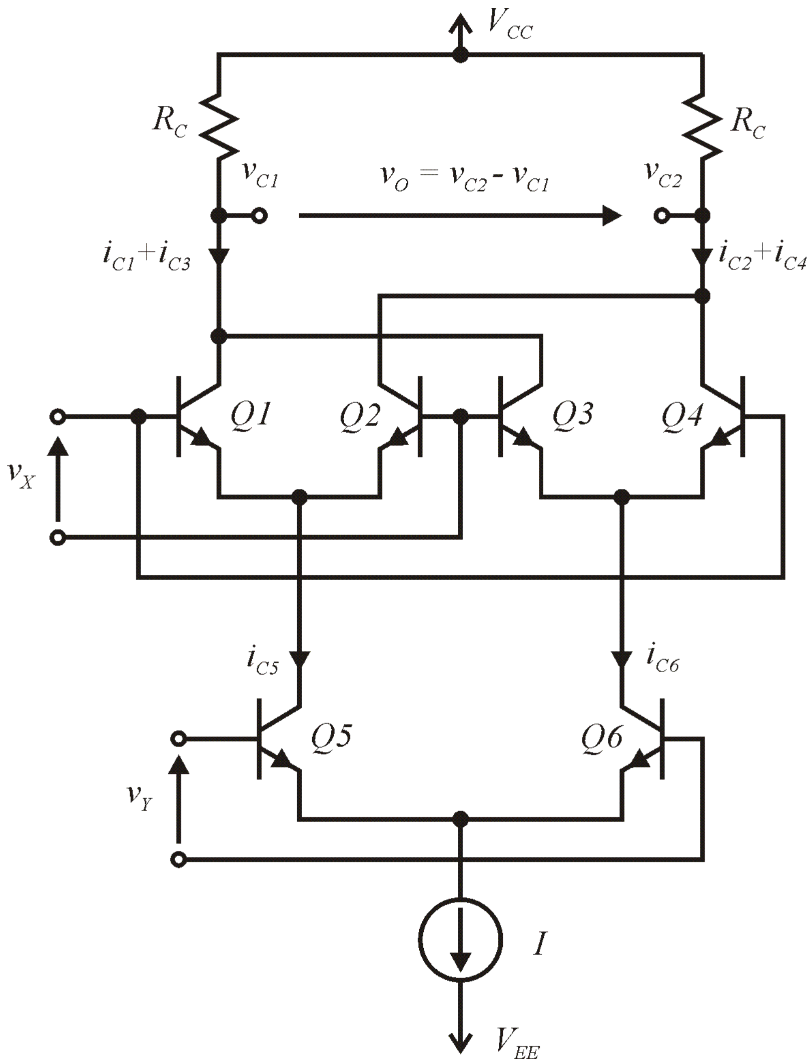


Fig. 5. Gilbert multiplier cell.

$$i_O = (i_{C1} + i_{C3}) - (i_{C2} + i_{C4}) = (i_{C1} - i_{C2}) - (i_{C4} - i_{C3})$$

$$= i_{C5} \tanh\left(\frac{v_X}{2V_T}\right) - i_{C6} \tanh\left(\frac{v_X}{2V_T}\right)$$

$$= (i_{C5} - i_{C6}) \tanh\left(\frac{v_X}{2V_T}\right)$$

$$= I \tanh\left(\frac{v_X}{2V_T}\right) \tanh\left(\frac{v_Y}{2V_T}\right)$$

$$= \Big|_{v_X \ll 2V_T;\, v_Y \ll 2V_T} = I\left(\frac{v_X}{2V_T}\right)\left(\frac{v_Y}{2V_T}\right)$$

$$\tanh(x) \approx x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \ldots$$

$$v_O = v_{C2} - v_{C1}$$

$$= V_{CC} - (i_{C2} + i_{C4})R_C - (V_{CC} - (i_{C1} + i_{C3})R_C)$$

$$= (i_{C1} + i_{C3})R_C - (i_{C2} + i_{C4})R_C = i_O R_C$$

$$\approx I\left(\frac{v_X}{2V_T}\right)\left(\frac{v_Y}{2V_T}\right)R_C = K_V v_X v_Y$$

Multiplier – increasing the linearity range for the Y input

For the circuits shown below, assuming that the v_Y signal passes to the emitters of the transistors without reducing the value, the collector currents are equal to:

$$i_{C5} = \frac{I}{2} + \frac{v_Y}{R_Y} \qquad i_{C6} = \frac{I}{2} - \frac{v_Y}{R_Y} \qquad i_{C5} - i_{C6} = 2\frac{v_Y}{R_Y} \qquad i_O = (i_{C5} - i_{C6})\tanh\left(\frac{v_X}{2V_T}\right) = 2\frac{v_Y}{R_Y}\tanh\left(\frac{v_X}{2V_T}\right) = \bigg|_{v_X \ll 2V_T} = \frac{v_X}{V_T}\frac{v_Y}{R_Y}$$

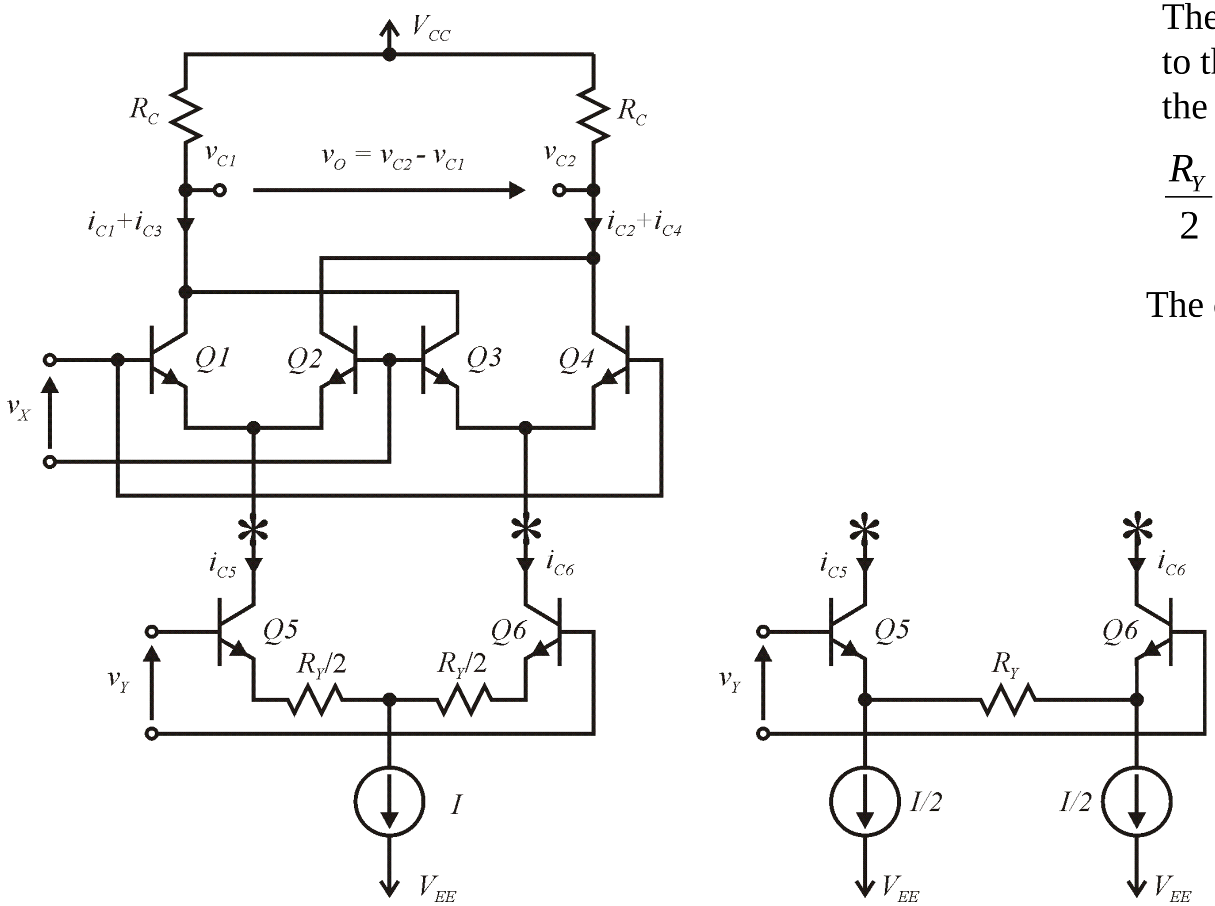


Fig. 6. Increasing the linearity range for the Y input of the Gilbert cell.

The condition of the 1: 1 signal transition to the Q5 and Q6 emitters (derived from the small signal *T* model of BJT):

$$\frac{R_Y}{2} >> r_{e5,6} \Rightarrow \frac{R_Y}{2} >> \frac{V_T}{I/2} \Rightarrow R_Y >> 4\frac{V_T}{I}$$

The condition of not draining the I / 2 current:

$$\frac{I}{2} > \left|\frac{v_Y}{R_Y}\right| \Rightarrow |v_Y| < \frac{1}{2} IR_Y$$

Gilbert multiplier - increasing the linearity range for both inputs

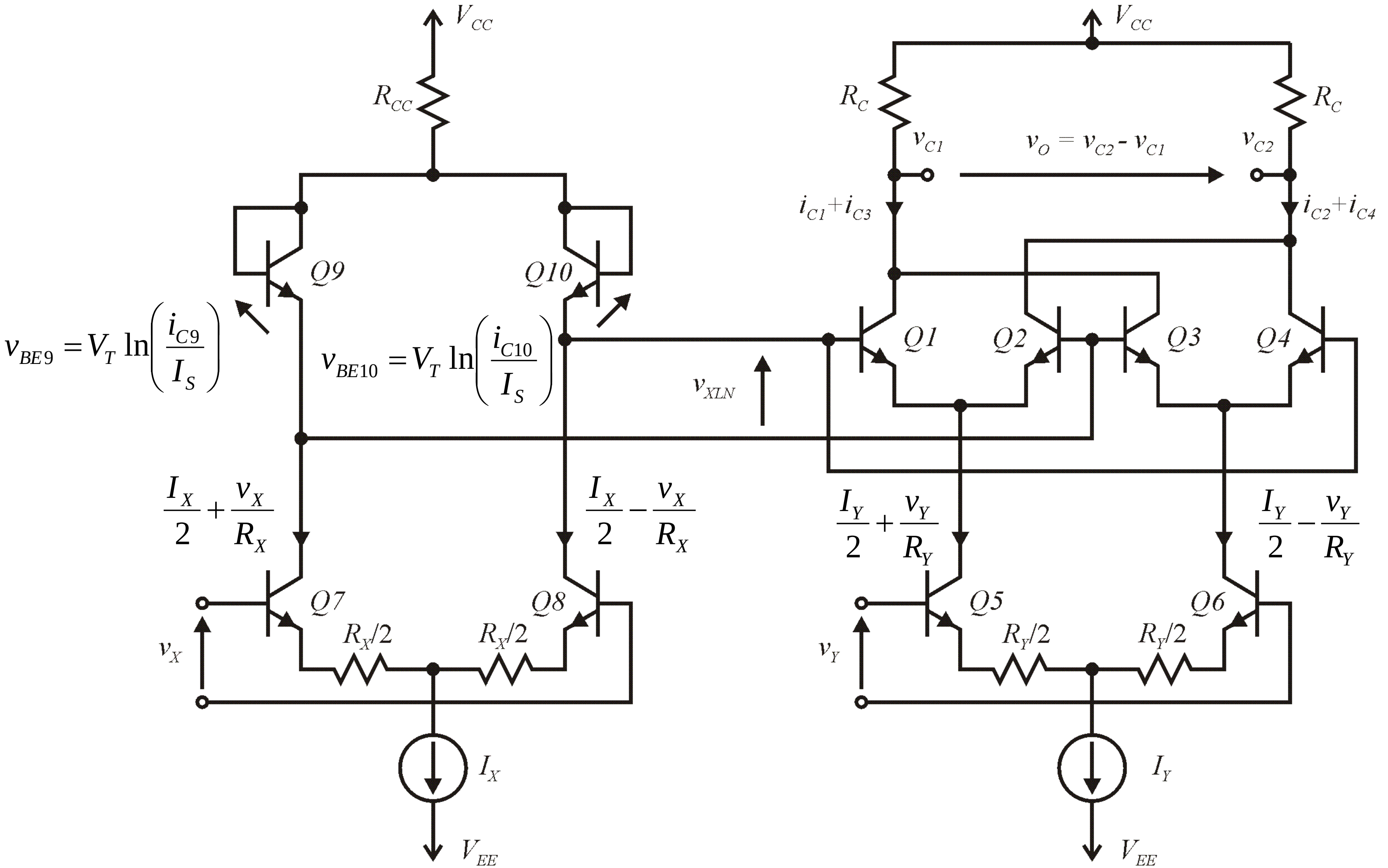


Fig. 7. Gilbert multiplier with both inputs linearisation.

For simplicity, α=1 was used, so the collector currents are equal to the emitter currents. It was assumed that the transistors are identical in pairs. The base-emitter voltage can be estimated as:

$$i_C = I_S e^{\frac{v_{BE}}{V_T}} \Rightarrow v_{BE} = V_T \ln \frac{i_C}{I_S}$$

For voltage designated as v_{XLN}, one can arrange Kirchhoff's voltage laws:

$$v_{XLN} = v_{BE1} - v_{BE2} = v_{BE9} - v_{BE10} \qquad v_{XLN} = v_{BE4} - v_{BE3} = v_{BE9} - v_{BE10}$$

So:

$$V_T \ln \frac{i_{C1}}{I_{S1}} - V_T \ln \frac{i_{C2}}{I_{S2}} = V_T \ln \frac{i_{C9}}{I_{S9}} - V_T \ln \frac{i_{C10}}{I_{S10}} \Rightarrow \ln \frac{i_{C1}}{i_{C2}} = \ln \frac{i_{C9}}{i_{C10}} \Rightarrow \frac{i_{C1}}{i_{C2}} = \frac{i_{C9}}{i_{C10}}$$

Substituting to the above equation and finding the difference in currents we get:

$$i_{C9} = I_X/2 + v_X/R_X \qquad i_{C10} = I_X/2 - v_X/R_X$$

$$i_{C1} - i_{C2} = i_{C5}\left(\frac{2}{I_X}\frac{v_X}{R_X}\right)$$

Similarly, for Q3 and Q4 pair their current difference can be defined as:

$$i_{C4} - i_{C3} = i_{C6}\left(\frac{2}{I_X}\frac{v_X}{R_X}\right)$$

Hence, the output current being the difference of the currents of the respective pairs is equal to:

$$i_O = (i_{C1} + i_{C3}) - (i_{C2} + i_{C4}) = (i_{C1} - i_{C2}) - (i_{C4} - i_{C3}) = i_{C5}\left(\frac{2}{I_X}\frac{v_X}{R_X}\right) - i_{C6}\left(\frac{2}{I_X}\frac{v_X}{R_X}\right)$$

$$= (i_{C5} - i_{C6})\left(\frac{2}{I_X}\frac{v_X}{R_X}\right) = \left(2\frac{v_Y}{R_Y}\right)\left(\frac{2}{I_X}\frac{v_X}{R_X}\right) = v_X v_Y \frac{4}{I_X R_X R_Y}$$

The above relationship is limited for input voltages that zeroes one of the lower differential pair currents, i.e. for input voltages in the range:

$$\frac{I_X}{2} > \left|\frac{v_X}{R_X}\right| \Rightarrow |v_X| < \frac{1}{2} I_X R_X \qquad \frac{I_Y}{2} > \left|\frac{v_Y}{R_Y}\right| \Rightarrow |v_Y| < \frac{1}{2} I_Y R_Y$$

The output voltage, similarly to previous circuits, can be determined as:

$$v_O = v_{C2} - v_{C1} = V_{CC} - (i_{C2} + i_{C4})R_C - (V_{CC} - (i_{C1} + i_{C3})R_C)$$

$$= (i_{C1} + i_{C3})R_C - (i_{C2} + i_{C4})R_C = i_O R_C = v_X v_Y \frac{4}{I_X R_X R_Y} R_C = K_V v_X v_Y$$

$$K_V = \frac{4R_C}{I_X R_X R_Y}$$

Differential current converter circuit

For the circuit as shown in the scheme, current Kirchhoff's law in the input nodes of operational amplifier *A* can be written as:

$$\frac{V_{CC}-v_{-}}{R_{C1}}+\frac{v_O-v_{-}}{R_{F1}}=i_1 \qquad \frac{V_{CC}-v_{+}}{R_{C2}}=\frac{v_{+}}{R_{F2}}+i_2$$

If we assume the equality of resistors

$R_{C1}=R_{C2}=R_C$ and $R_{F1}=R_{F2}=R_F$

and ideal amplifier *A*, then the output voltage is equal to:

$$v_O=(i_1-i_2)R_F$$

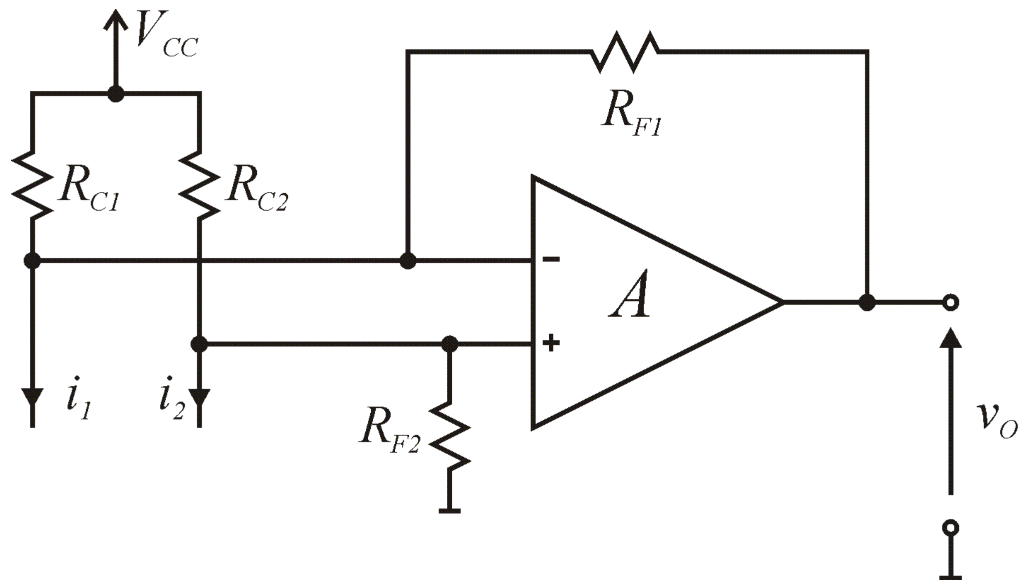


Fig. 8. Differential current to single output voltage converter.

$$v_{OUT} = i_O R_F = v_X v_Y \frac{4}{I_X R_X R_Y} R_F \qquad K_V = \frac{4R_F}{I_X R_X R_Y} \qquad v_O \neq i_O R_C$$

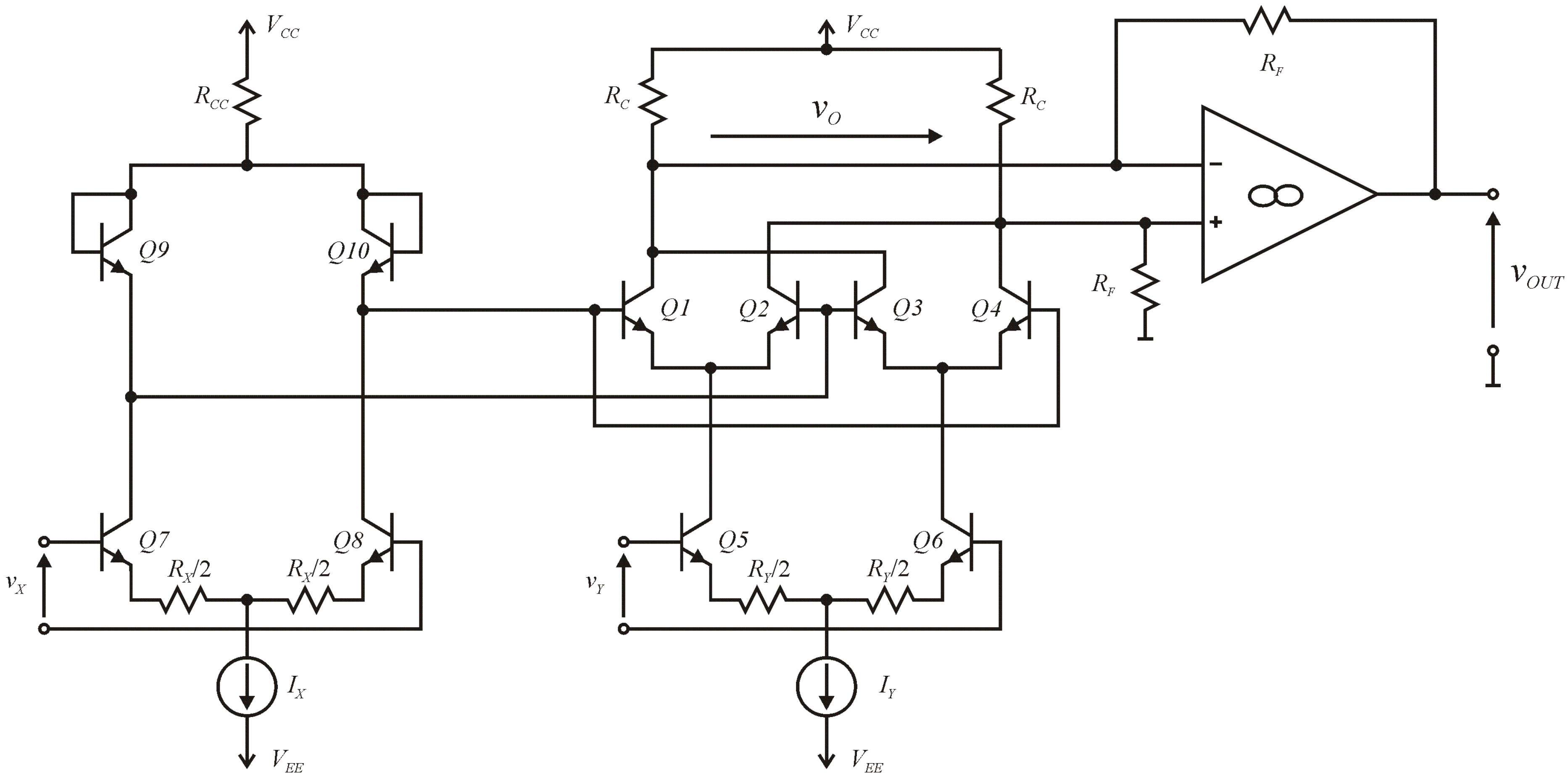


Fig. 9. Linearized four-quadrant Gilbert multiplier with voltage output.

Implementation of selected non-linear functions

- Division
- Square up
- Square root
- Logarithm
- Exponential circuit

Ideal Operational Amplifier virtual short circuit rule

For an ideal OA, differential gain A_D tends to infinity. Let's assume that OA works in negative feedback loop as e.g. in the picture below.

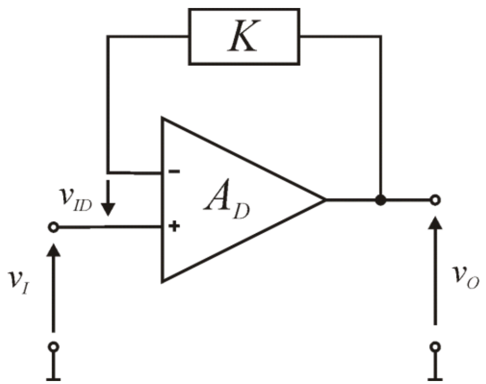
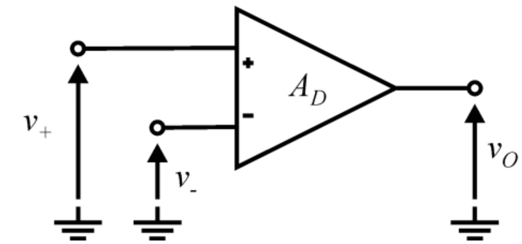


Fig. 11. Operational amplifier with negative feedback loop K.



$$v_O = A_D\left(v_+ - v_-\right) \qquad v_{ID} = v_+ - v_-$$

Fig. 10. Operational Amplifier (OA) symbol.

Circuit equations for the above case:

$$v_O = A_D v_{ID} = A_D\left(v_I - v_O K\right) \Rightarrow v_O\left(1 + A_D K\right) = A_D v_I \Rightarrow v_O = v_I \frac{A_D}{1 + A_D K}\Big|_{A_D \to \infty} = v_I / K$$

And what is the value of differential input voltage?

$$v_{ID} = \frac{v_O}{A_D} = \frac{v_I \frac{A_D}{1 + A_D K}}{A_D} = v_I \frac{1}{1 + A_D K}\Big|_{A_D K \to \infty} = 0$$

In an ideal OA with negative feedback loop the voltage difference between the inputs tends to zero, which in practice means a "virtual" short circuit between the amplifier's inputs

Division

Assuming negative feedback loop input voltage v_Y is limited to positive values. For ideal OA:

$$i_{R1} = \frac{v_X}{R_1} = \frac{-K_V v_Y v_O}{R_2} \Rightarrow v_O = -\frac{R_2}{K_V v_Y}\frac{v_X}{R_1} = -\frac{R_2}{K_V R_1}\frac{v_X}{v_Y}$$

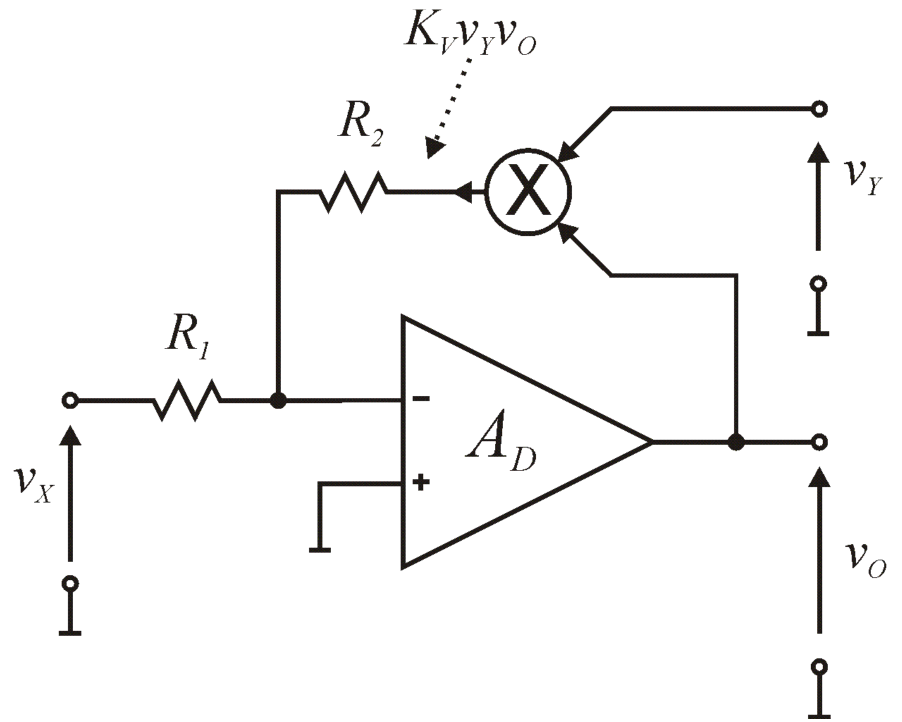


Fig. 12. Division circuit.

Square up

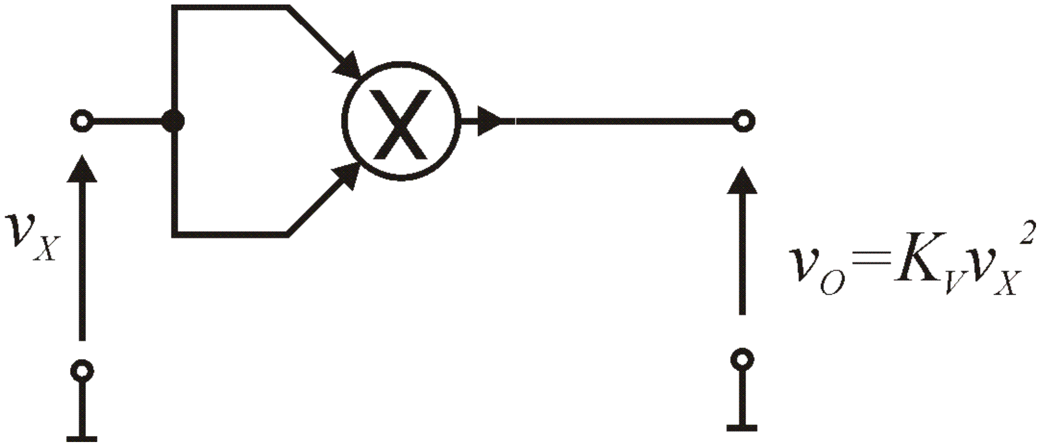


Fig. 13. Square up circuit with the use of multiplier.

Square root

Assuming the negative feedback loop the output voltage v_O is limited to positive values, so the input voltage must be negative! Current equation assuming ideal OA:

$$i_{R1} = -\frac{v_X}{R_1} = \frac{K_V v_O^{\ 2}}{R_2} \quad \Rightarrow \quad v_O = \sqrt{-v_X \frac{R_2}{K_V R_1}}$$

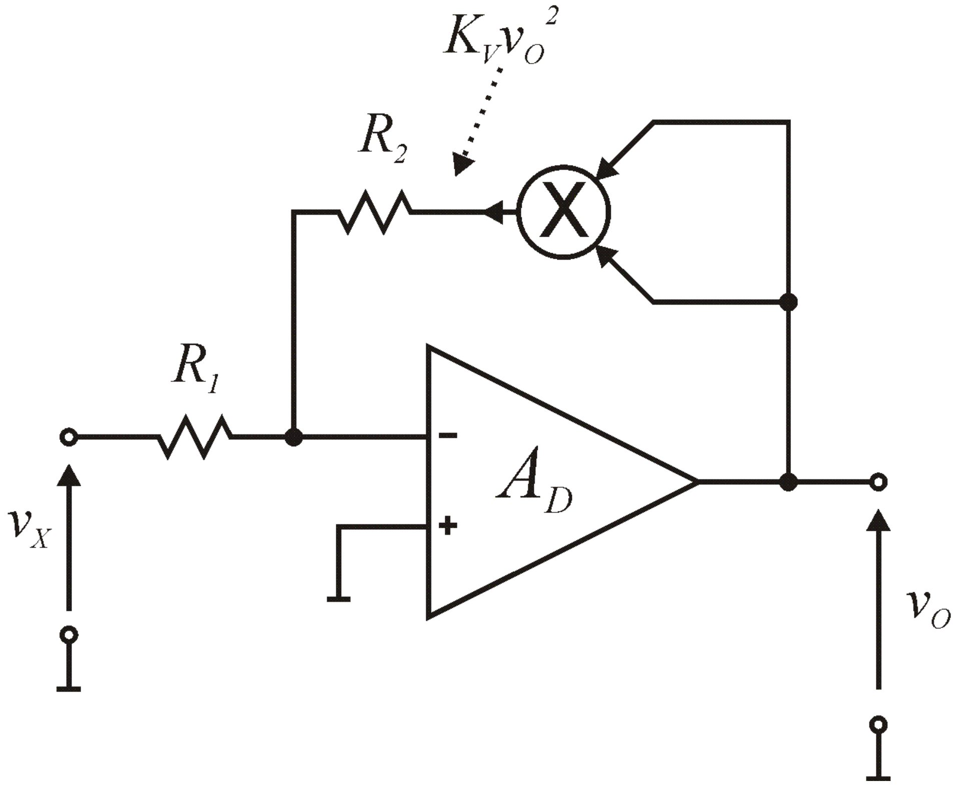


Fig. 15. Square root circuit.

Logarithm

Assuming negative feedback loop and positive collector current input voltage v_X is limited to positive values. For ideal OA:

$$i_{R1} = \frac{v_X}{R_1} = i_C = I_S e^{\frac{v_{BE}}{V_T}} = I_S e^{\frac{-v_O}{V_T}} \Rightarrow \ln\left(\frac{v_X}{I_S R_1}\right) = \frac{-v_O}{V_T} \Rightarrow v_O = -V_T \ln\left(\frac{v_X}{I_S R_1}\right)$$

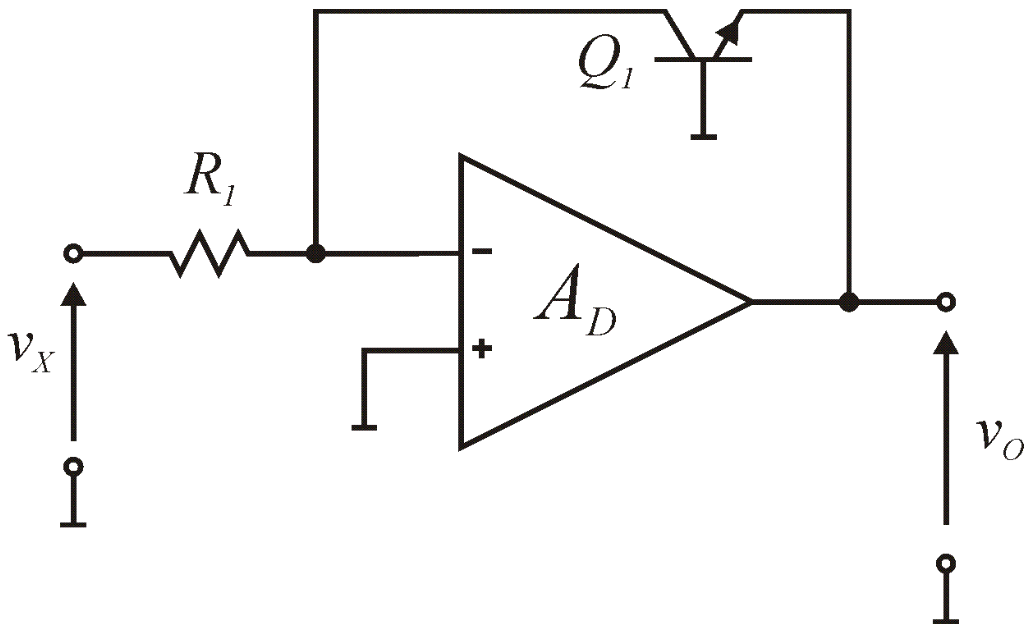


Fig. 16. Logarithm circuit.

Exponential circuit

Assuming negative feedback loop and positive collector current input voltage v_X is limited to positive values. For ideal OA:

$$i_{R2} = \frac{v_O - V_P}{R_2} = i_C = I_S e^{\frac{v_{BE}}{V_T}} = I_S e^{\frac{v_X}{V_T}} \Rightarrow v_O = R_2 I_S e^{\frac{v_X}{V_T}} + V_P$$

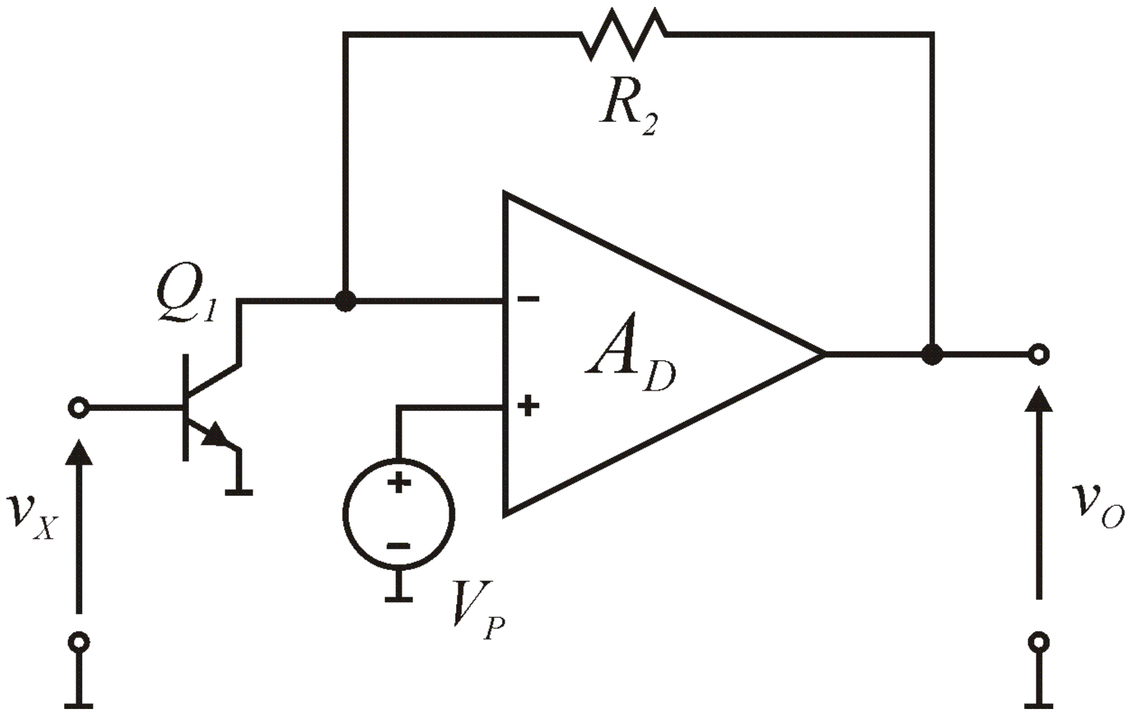


Fig. 17. Exponential circuit.

Due to the necessity to force the NPN transistor to work in the active normal range it is necessary to apply polarizing voltage V_P, $v_{CE} = V_P$.